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Strings on orbifold lines

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Abstract: The orbifold lines IIA/$\mathcal{I}_8$ and IIB/$\mathcal{I}_8(-1)^{F_L}$ possess BPS discrete torsion variants which carry fundamental string (NSNS) charge. We show that these variants are actually classified by an integral electric field F from the twisted RR sector, and compute their tension and NSNS charge as a function of F. The analysis employs equivariant K-theory and the string creation phenomenon. The K-theory results demonstrate the corrections to cohomology in the case of torsion; it is found that 8 units of F are invisible at transverse infinity for IIA, and correspondingly 16 units for IIB.

Keywords: p-branes, Superstring Vacua.

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Contents

1. Introduction and summary

Some string theory backgrounds are known to possess discrete variants. In particular, certain orbifolds have discrete torsion variants [1, 2], which can be thought of as arising from finite discrete fluxes of the NSNS two form field B concentrated at the fixed points. More recently it has been realized that discrete RR fluxes can also give rise to orbifold variants. In particular, the orbifolds $\mathbb{R}^8/\mathcal{I}_8$ and $\mathbb{R}^8/\mathcal{I}_8(-1)^{F_L}$ of type-IIA and type-IIB string theory, respectively, admit a set of variants corresponding to the possible discrete RR fluxes at infinity. These fluxes were originally classified in integer cohomology, and found to correspond to torsion elements [3].

It has also been proposed recently that RR fields take values in K-theory rather than integer cohomology [4]. It is not surprising that RR charges, or D-branes, are described by K-theory [5, 6], since one needs to specify a gauge bundle on the D-brane, as well as the homology cycle it wraps, to fully characterize it. On the other hand D-branes are sources for RR fields, so it is natural that the latter should be valued in K-theory as well. Since K-theory differs from cohomology only in discrete torsion, it becomes relevant for the study of the above types of variants.

This paper was partly motivated by the study of discrete torsion variants of orientifolds [3](see also [7, 8, 9]). In this approach, the choice of orientifold projection leading to an SO or Sp gauge group corresponds to a discrete torsion of the B field. The objective of [3] was to list all possible variants by identifying the background fields which allow for discrete torsion. However, the analysis there relied on cohomology, and some questions were left unsettled. In particular, questions were raised about the variants of the above orbifold lines (which are related to orientifolds by dualities), which were denoted $OF1_A$ (IIA on $\mathbb{R}^8/\mathcal{I}_8$) and $OF1_B$ (IIB on $\mathbb{R}^8/\mathcal{I}_8(-1)^{F_L}$). These were found to have some variants which carry fundamental string (NSNS) charge (hence the 'F' in their names), but a general formula relating the RR fluxes to the NSNS charge was not obtained. The main purpose of this paper is to derive precisely such a relation.

Brane	D6	D4	D2
Tension jump	1/16	1/4	1

Table 1: Tension jumps for the orbifold line $OF1_A$ after intersecting with various D-branes (in fundamental string tension units).

We begin in section 2 by reviewing the results of [3] on the discrete torsion variants of the above orbifolds. For $OF1_A$, the RR fluxes were enumerated in reduced cohomology by $H_R^{\text{even}}(\mathbb{RP}^7) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,[1] and were interpreted as arising from intersections with a fractional D2-brane, D4-brane and D6-brane. The intersection point serves as a "domain wall" on the orbifold line, across which its tension, and accordingly its fundamental string charge, can jump. The tension jump for the three cases is given in table 1. We will review the derivation of these results, and explain how they are consistent with T-duality. The review also includes some basic examples of discrete torsion for completeness.

We then explain how re-interpreting RR fields in K-theory corrects this picture. In particular, for the IIA orbifold, reduced cohomology is replaced by the reduced K-theory group $\widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) = \widetilde{K}(\mathbb{RP}^7) = \mathbb{Z}_8$. The "enhancement" of the discrete torsion group from cohomology to K-theory actually has an interesting physical interpretation. While in cohomology it appears that two fractional Dp-branes have a trivial discrete torsion, in K-theory we see that two fractional Dp-branes correspond to the same discrete torsion as one D$(p-2)$-brane. So, for example, adding two D4's gives a D2 class instead of a trivial one. K-theory also allows us to make a distinction between RR fields at infinity, which take values in $\widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) = \mathbb{Z}_8$, and RR fields in the total (singular) transverse space, which take values in $\widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0}) = \mathbb{Z}$.[2] The

[1]Reduced cohomology is defined as $H_R^* \equiv H^*/H^0 = H^*/\mathbb{Z}$.

[2]The latter space is usually not analyzed by ordinary cohomology ("ordinary" here is "non equivariant"), since it is singular.

two groups are related, and as a consequence the generator x of the latter group satisfies that $8x$ is invisible at infinity.[3] The analogous groups in the IIB orbifold are $K_{\pm}^{-1}(\mathbf{S}^{8,0}) = \mathbb{Z}_{16}$ and $K_{\pm}^{-1}(\mathbb{R}^{8,0}) = \mathbb{Z}$, so 16 times the generator of the latter group is trivial at infinity.

In section 3 we analyze in detail the orbifolds $\mathbb{R}^8/\mathcal{I}_8$ and $\mathbb{R}^8/\mathcal{I}_8(-1)^{F_L}$ for both type IIA and type IIB. We determine both the perturbative closed string spectrum and the D-brane spectrum of these theories. In particular, it is shown that the $\mathcal{I}_8$ orbifold of IIA ($OF1_A$) and the $\mathcal{I}_8(-1)^{F_L}$ orbifold of IIB ($OF1_B$) contain a massless vector field A in the twisted RR sector. Since this field lives in two dimensions it has no dynamics, but it does allow for charged point-like defects, across which the scalar field strength $F = *dA$ jumps by a discrete amount. These defects are precisely the fractional Dp-branes of $OF1_A$ and $OF1_B$ (and a non-BPS D-particle in $OF1_B$). The corresponding jump in the value of F is given by

$$\Delta F_p = 2^{-p/2} \tag{1.1}$$

for the fractional Dp-branes, and $\Delta F_{\tilde{0}} = \sqrt{2}$ for the non-BPS D-particle.

These jumps can now be identified with the integral charges in $\widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0})$ (and $K_{\pm}^{-1}(\mathbb{R}^{8,0})$) above, since both are generated by intersections with fractional D-branes. In the IIA case, the generator x of $\widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0})$ corresponds to the fractional D6-brane, which carries a charge $\Delta F = 1/8$, and the element $8x$ is equivalent to the fractional D-particle, which has a charge 1, and is invisible at infinity. In the IIB case, the generator corresponds to the fractional D7-brane (with charge $\sqrt{2}/16$), and the element $16x$ is the non-BPS D-particle (with charge $\sqrt{2}$). This is an example of how K-theory correlates different RR fields; in this case the different fields are the twisted sector and bulk RR fields.

Section 4 contains our main result, namely the tension and NSNS charge of the orbifold lines as a function of the twisted sector scalar field F. We find that

$$Q(F) = 4F^2 - \frac{1}{16}\,, \tag{1.2}$$

which together with (1.1) is consistent with the earlier results (table 1). The constant $Q(0) = -1/16$ was computed in [10] based on the term $\int B \wedge Y_8(R)$ in the effective type-IIA action [11] (see section 3 for more detail, and [12] for related work). This value ties well with the known charges of objects related by dualities to orbifold lines; both the orientifold line and the $OM2^-$ have the same tension[4] (for the $OM2$ see [9]). Our concern here is to derive the F dependence of $Q(F)$, that is to derive $\Delta Q(F) = Q(F) - Q(0)$.

The relation (1.2) is first obtained from the low-energy effective action. The kinetic term for F gives a vacuum energy per unit length, i.e. a tension, proportional

[3] Invisible at infinity means that it is mapped to zero by the natural mapping into $\widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{8,0})$.
[4] $OM2^-$ denotes $M/\mathcal{I}_8$.

to F^2. The correct normalization is determined by the sphere closed string diagram with two insertions of F, which is related to a simple one-loop open string amplitude that computes the supersymmetry index of the internal ($\mathbb{R}^8/\mathbb{Z}_2$) CFT.

In an alternative approach, we use the fact that pairs of fractional D-branes in these orbifolds are linked, and therefore exhibit the phenomenon of string creation [13, 14, 15]. We consider specifically the case of two fractional D-particles in $OF1_A$. This system is similar to the D0-D8 system in ten dimensions, in which a single string is created when the branes cross. The number of strings created can be understood in terms of the supersymmetric index of the 0-8 open string. Our case is a bit more subtle, in that the particle and domain wall are both D-particles, but the number of strings which are created is still given by the supersymmetry index, which in this case is $n = 8$. The number of strings ending on a single D-particle, and therefore the tension jump of the orbifold line, can then be determined from a consistency condition for the process of exchange, and the result is consistent with (1.2).

Note that the constant $n = 8$ is determined in two seemingly unrelated ways — the first from K-theory, and the second as an open string index. We argue in the same section that the two are actually related, by considering a source term for $B_{\rm NS}$ in the type-IIA action, which depends on a certain self-intersection number. Here we assume that $\mathbb{R}^8/\mathbb{Z}_2$ has a singular (shrunk) 4-cycle C, such that the fractional D-particle is a D4-brane wrapping C, and that its self-intersection number must be ± 8. On one hand the self intersection number is geometric and determined by K-theory, but on the other hand when two such D4-branes are exchanged the number of strings created is exactly the self-intersection number, so the two approaches are connected.

The case of $OF1_B$ is similar to $OF1_A$, and the reader may concentrate his / her attention on the latter in a first reading. Collecting the results for the $OF1_B$ we find that the K group at infinity is $\mathbb{Z}_{16}$, replacing the cohomological $(\mathbb{Z}_2)^4$ which arises from intersections with the fractional D7, D5, D3, and D1. The normalization for F is that the fractional D7 (the generator) has charge $\sqrt{2}/16$, while the non-BPS D-particle, which is invisible at infinity, has charge $\sqrt{2}$. The formulas (1.1) and (1.2) are the same for both orbifold lines.

2. RR discrete torsion

2.1 Discrete torsion

It was observed in [1] that for certain orbifolds one can multiply the twisted components of the string partition function by phases, while maintaining consistency, and in particular modular invariance. For an orbifold M/Γ, where is M is the covering space on which a discrete group Γ acts, the possibilities for adding such phases were found to be classified by the second cohomology group of Γ with U(1) coefficients $H^2(\Gamma, \mathrm{U}(1))$. The simplest example is $H^2(\mathbb{Z}_n \times \mathbb{Z}_n, \mathrm{U}(1)) = \mathbb{Z}_n$.

Alternatively, discrete torsion can be viewed as arising from *spacetime* cohomology rather than *group* cohomology (see [16] for a rigorous treatment). The phases are due to a non-trivial topology of the NSNS B field. The contribution of B to the functional integral is through the term $\exp(\frac{i}{2}\int B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}) = \exp(\frac{i}{2}\int B^{\text{induced}})$, and therefore in the presence of a flat B field all world-sheets in the same spacetime homology class will receive the same contribution. Thus, by definition, the phases are classified by the cohomology $H^2(M/\Gamma, \mathrm{U}(1))$. This can be further simplified by observing that the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathrm{U}(1) \to 0$ identifies the components of $H^2(M/\Gamma, \mathrm{U}(1))$ with the discrete torsion part of $H^3(M/\Gamma, \mathbb{Z})$. It is natural to identify this class with the class of the field strength $[H] = [dB] \in H^3(M/\Gamma, \mathbb{Z})$.

By analogy, if we replace the world-sheet of the fundamental string with the world-volume of a D-brane, we expect that the full non-perturbative partition function will have additional discrete torsion degrees of freedom. Namely, for any RR p-form field strength $G^{(p)}$, one expects to see variants classified by $H^p(M/\Gamma, \mathbb{Z})$. For type IIA this implies $H^{\text{even}}(M/\Gamma, \mathbb{Z})$ variants, while for IIB $H^{\text{odd}}(M/\Gamma, \mathbb{Z})$ variants are expected. We will soon see that this picture is in fact corrected when we reinterpret RR fields, and in particular RR discrete torsion, in K-theory.

The simplest example of discrete torsion corresponds to a vector field on $\mathbb{R}^d/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as $x \to -x$. In this case $H^2((\mathbb{R}^d \backslash \{0\})/\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$, so a non-trivial configuration for the gauge field is possible. Such a configuration would be detected by sending a charged particle on a closed path from a point to its antipode, and observing a phase of -1, even though the field strength is locally zero. In other words we have a flat circle bundle $(\mathbb{R}^d \times \mathbf{S}^1)/\mathbb{Z}_2$, with the $\mathbb{Z}_2$ acting on the circle $0 \leq \theta \leq 1$ by $\theta \to \theta + 1/2$.

2.2 Previous results from cohomology

Let us review the results obtained in [3] regarding the classification and properties (namely the tension and NSNS charge) of the orbifolds $OF1_A$ and $OF1_B$. The relevant cohomologies for $OF1_A$ are $H^{\text{even}}(\mathbb{RP}^7, \mathbb{Z}) = H^0 \oplus H^2 \oplus H^4 \oplus H^6 = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, corresponding respectively to the RR fields $G^{(0)}, G^{(2)}, G^{(4)}$, and $G^{(6)}$. In $OF1_B$ all the RR forms are twisted in the sense that they switch sign upon inversion. The relevant cohomologies are therefore $H^{\text{odd}}(\mathbb{RP}^7, \widetilde{\mathbb{Z}}) = \widetilde{H}^1 \oplus \widetilde{H}^3 \oplus \widetilde{H}^5 \oplus \widetilde{H}^7 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, corresponding to $G^{(1)}, \ldots, G^{(7)}$, where $\widetilde{\mathbb{Z}}$ is the twisted bundle (or sheaf) of integers on $\mathbb{RP}^7$.

Discrete torsion fluxes can be interpreted as arising from fractional branes. In order to create a $\mathbb{Z}_2$ flux for a p-form field strength $G^{(p)}$ one takes a D$(8-p)$-brane which is its source, and moves it onto the orbifold where the D-brane can split into two halves. By sending the two halves far away in opposite directions one constructs a new orbifold which carries the required discrete torsion.

According to this interpretation $OF1_A$ can be intersected by D8, D6, D4 and a D2. The D8 is special as it changes the cosmological constant. It is represented by the $\mathbb{Z}$ factor in $H^{\text{even}}(\mathbb{RP}^7, \mathbb{Z})$. The $OF1_B$ on the other hand can be intersected by the D7, D5, D3 and D1. Note that all of the above intersection configurations are supersymmetric, and that in all cases the flux from a jump in the fundamental string charge can be carried away by fields in the world-volume of the brane.

Let us see how the jump in fundamental string charge was determined in some cases by an ad-hoc series of dualities.

- The intersection of the $OF1_A$ with a D4-brane lifts in M theory to an intersection in 11d of the OM2 with an M5-brane. This is known to cause a jump of $+1/4$ in the membrane charge [9], which upon compactification becomes the fundamental string charge.

- The intersection of the $OF1_B$ with a D5-brane is S-dual to an intersection of the orientifold line with an NS5-brane. This intersection is known to change an $O1^-$ (SO projection) into an $O1^+$ (Sp projection), causing a jump of $1/16 - (-1/16) = +1/8$ in the D-string charge, which is mapped into the F-string charge in the original picture.

- The intersection with the D3 can be analyzed in the same way: after S-duality one gets an orientifold line $O1^-$ which is transformed by the D3-brane into an $\widetilde{O1}^-$, which is an $O1^-$ with an extra half D1-brane. Therefore the jump here is by $+1/2$.

- In agreement with the trend, one also expects the intersection with a D6 to lead to a tension jump of $+1/16$. For that one needs to assume that the tension of the "bare" $OF1_A$ is $-1/16$ (see section 3). The intersection with a D6 is equivalent to a discrete torsion for the vector field of type IIA. When lifted to M theory, it is described by a smooth manifold $(\mathbb{R}^8 \times \mathbf{S}^1)/\mathbb{Z}_2$ (as explained in the example in previous subsection), and is not expected to carry any charges.

- For the D2-brane it was found by a less direct method that the jump is $+1$.

We summarize the results in table 2. The pattern that emerges can be explained by T-duality. Consider the orbifold to act on a torus, that is $\mathbf{T}^8/\mathbb{Z}_2$, rather than $\mathbb{R}^8/\mathbb{Z}_2$ as we did so far. In this case the number of fixed points on the torus is $2^8 = 256$.

Brane	D6	D5	D4	D3	D2
Tension jump, ΔQ	1/16	1/8	1/4	1/2	1

Table 2: Tension jumps for the orbifold lines $OF1_{A,B}$ after intersecting with various D-branes (in fundamental string tension units).

When we perform a T-duality in one of the directions of the torus, the 256 $OF1_A$ lines are replaced by 256 $OF1_B$ lines (or vice versa), and D-branes are replaced according to the usual rule. Since the total tension jump (the sum of jumps for all 256 lines) is invariant under T-duality, and since a Dp-brane intersects 2^p lines, it follows that the jump on a single orbifold line is proportional to 2^{-p}, in agreement with table 2.

2.3 K-theory corrections

It has become clear in the past year that RR fields and charges take values in K-theory, rather than integer cohomology. This was suggested by the fact that the states which carry RR charges, namely D-branes, possess an internal structure which includes a gauge bundle [5], and from the conjecture that a brane and an anti-brane which carry isomorphic bundles annihilate into the vacuum [17, 6]. It follows that D-branes are classified by isomorphism class groups of gauge bundles, i.e. K-theory. In particular, D-branes on a smooth space X are classified by the compactly supported K-theory groups $K^{-1}_{cpct}(X)$ and $K_{cpct}(X)$ in type-IIA and type-IIB string theory, respectively.[5] Moreover, since D-branes are sources of RR fields, the latter also take values in K-theory [4]. In particular, source free RR fluxes in a space Y are classified by $K(Y)$ in type IIA, and by $K^{-1}(Y)$ in type IIB.

K-theory is also well-suited to handle singular spaces such as orbifolds and orientifolds, and it is in these cases where K-theory generally differs from cohomology. In orbifold backgrounds RR charges and fields take values in *equivariant* K-theory. In particular , for orbifolds of the type $\mathbb{R}^n/\mathcal{I}_n$ the relevant groups are $K_{\mathbb{Z}_2}$ and $K^{-1}_{\mathbb{Z}_2}$, and for orbifolds of the type $\mathbb{R}^n/\mathcal{I}_n(-1)^{F_L}$ the corresponding groups are $K_\pm$ and $K^{-1}_\pm$.

For the orbifolds $OF1_A$ and $OF1_B$ K-theory differs from cohomology in two ways. First, RR fluxes at infinity take values in[6]

$$\begin{aligned} \widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) &= \mathbb{Z}_8 \quad \text{for } \; OF1_A \\ K^{-1}_\pm(\mathbf{S}^{8,0}) &= \mathbb{Z}_{16} \quad \text{for } \; OF1_B\,. \end{aligned} \tag{2.1}$$

The discrete torsion corresponding to the fields $G^{(8-p)}$, with $1 \leq p \leq 7$, is different from cohomology. Recall that in cohomology the torsion subgroup was $(\mathbb{Z}_2)^3$ and $(\mathbb{Z}_2)^4$ for $OF1_A$ and $OF1_B$, respectively. The dimensions agree with K-theory, but the group structure is different. Let us denote the $\mathbb{Z}_2$ generator in the n'th cohomology group by $x^{(n)}$, where $n = 2, 4, 6$ for $OF1_A$, and $n = 1, 3, 5, 7$ for $OF1_B$. In cohomology these satisfy

$$2x^{(n)} = 0\,. \tag{2.2}$$

[5]To be precise, consistent (tadpole-free) RR charges in type IIB are actually given by the kernel of the natural map $K_{cpct}(X) \to K(X)$ (with the obvious generalization to type IIA).

[6]The notation $\mathbf{S}^{l,m}$ refers to the sphere at infinity in the space $\mathbb{R}^{l,m}$, which corresponds to $\mathbb{R}^{l+m}/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ inverts l coordinates. In particular $\mathbf{S}^{l,0} \sim \mathbb{RP}^{l-1}$, but the former notation reminds us that the $\mathbb{Z}_2$ acts on the bundles as well.

This can be understood physically by the fact that two fractional Dp-branes (with $p \leq 7$) can combine into a bulk Dp-brane, which can separate from the orbifold line, and thereby remove the flux from the $\mathbb{RP}^7$ at infinity. On the other hand, the above results suggest loosely that in K-theory

$$2x^{(n)} = x^{(n+2)} \tag{2.3}$$

(see the appendix for a more rigorous derivation). Physically this means that the RR flux at infinity due to two fractional Dp-branes ($p \leq 7$) is not trivial, but rather corresponds to the RR flux of a fractional D$(p-2)$-brane.[7] We could still recover a bulk D-brane, however, by combining two fractional D-branes with opposite twisted charge (corresponding to $x^{(n)}$ and $-x^{(n)}$) if they carry the same bulk charge.

The second K-theory correction appears in the spectrum of RR fields in the total transverse space, specifically when we compare fields which are compactly supported to the more general case. The former take values in

$$\begin{aligned} K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{8,0}) &= \mathbb{Z} \oplus \mathbb{Z} \quad \text{for } OF1_A \\ K^{-1}_{\pm,cpct}(\mathbb{R}^{8,0}) &= \mathbb{Z} \qquad\quad\ \ \text{for } OF1_B\,. \end{aligned} \tag{2.4}$$

Since the only compactly supported RR fields are associated with D-particles, it is clear that one of the integral fluxes in $OF1_A$ corresponds to $G^{(8)}$. As we shall see in the next section, the other integral flux, as well as the flux in $OF1_B$, corresponds to a twisted sector RR field F, under which all fractional D-branes (as well as a stable non-BPS D-particle in $OF1_B$) are charged. Both of these fluxes are absent on the $\mathbb{RP}^7$ at infinity (2.1); the 8-form cannot be supported on a 7-manifold, and the twisted sector field is supported only at the origin. On the other hand, the fluxes which were visible at infinity are absent here because they are not compact in the transverse space. If we relax the condition of compact support we get

$$\begin{aligned} \widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0}) &= \mathbb{Z} \quad \text{for } OF1_A \\ K^{-1}_{\pm}(\mathbb{R}^{8,0}) &= \mathbb{Z} \quad \text{for } OF1_B\,. \end{aligned} \tag{2.5}$$

[7]Two comments are in order here. First, for $OF1_A$ there is also an integral $G^{(0)}$ flux, which is apparent in the unreduced group $K_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) = \mathbb{Z} \oplus \mathbb{Z}_8$, and which agrees with unreduced cohomology. However, the cohomology generator satisfies $2x^{(0)} = \{G^{(0)} = 2\}$, whereas the K-theory generator satisfies $2x^{(0)} = \{G^{(0)} = 2\} + x^{(2)}$. The second comment is that torsion group structure in K-theory implies that each fractional Dp-brane carries $1/2$ unit of fractional D$(p-2)$-brane charge, which in turn requires an anomalous Dirac quantization condition for the gauge field on the p-brane

$$\int_{\mathbb{RP}^2 \subset \mathbb{RP}^7} \frac{F}{2\pi} = \frac{1}{2} + \mathbb{Z}\,.$$

The latter can be viewed as a manifestation of the difference between K-theory torsion and cohomology torsion.

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This is surprising at first, since one expects all the fluxes to contribute in this case, but clearly (2.5) is not the direct sum of (2.1) and (2.4). In fact, the three groups are part of a long exact sequence (appendix), which shortens to

$$
\begin{array}{ccccccccc}
0 \to & K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) & \to & K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{8,0}) & \stackrel{i}{\to} & \widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0}) & \to & \widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) & \to 0 \\
& \| & & \| & & \| & & \| & \\
& \mathbb{Z}' & & \mathbb{Z}' \oplus \mathbb{Z}_y & \xrightarrow{\times 8} & \mathbb{Z}_x & & \mathbb{Z}_8 &
\end{array}
\tag{2.6}
$$

for $OF1_A$, and to

$$
\begin{array}{ccccccc}
0 \to & K^{-1}_{\pm,cpct}(\mathbb{R}^{8,0}) & \stackrel{i}{\to} & K^{-1}_{\pm}(\mathbb{R}^{8,0}) & \to & K^{-1}_{\pm}(\mathbf{S}^{8,0}) & \to 0 \\
& \| & & \| & & \| & \\
& \mathbb{Z}_y & \xrightarrow{\times 16} & \mathbb{Z}_x & & \mathbb{Z}_{16} &
\end{array}
\tag{2.7}
$$

for $OF1_B$.[8] We have labeled the different integral fluxes according to how they map in the sequence. It follows that $\mathbb{Z}$ and $\mathbb{Z}'$ are identified with F and $G^{(8)}$, respectively.[9] The sequences show that the generator y of $\mathbb{Z}$ in the compactly supported group is related to the generator x of $\mathbb{Z}$ in the general group by

$$
\begin{aligned}
y &= 8x \quad \text{for } OF1_A \\
y &= 16x \quad \text{for } OF1_B\,.
\end{aligned}
\tag{2.8}
$$

The element y corresponds to the F-flux of a fractional (stable non-BPS) D-particle in $OF1_A$ ($OF1_B$), and is therefore trivial on the $\mathbb{RP}^7$ away from the orbifold line. On the other hand x maps to the generator of the torsion group on $\mathbb{RP}^7$, and therefore corresponds to the F-flux of a fractional D6-brane (D7-brane) in $OF1_A$ ($OF1_B$). It then follows that the element $2^k \cdot x$ (for $k \leq 3$) of the non-compactly supported group corresponds to the F-flux of the fractional D$(6-2k)$-brane (D$(7-2k)$-brane). The relative F-fluxes of the fractional Dp-branes are therefore given by

$$
F \propto 2^{-p/2}\,.
\tag{2.9}
$$

We will make this more precise in the following section.

3. Perturbative orbifold lines

In this section we analyze in detail the perturbative $\mathbb{Z}_2$ orbifold lines $\mathbb{R}^{1,1} \times \mathbb{R}^8/\mathcal{I}_8$ and $\mathbb{R}^{1,1} \times \mathbb{R}^8/\mathcal{I}_8(-1)^{F_L}$ for both type-IIA and type-IIB string theory. The type-IIA

[8] In the first sequence the first two K groups are actually equal to their reduced versions.

[9] Note that the flux of $G^{(8)}$ ($\mathbb{Z}'$) is present in $K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{8,0})$, but absent in $\widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{8,0})$. This should follow from an argument analogous to the one given in [4] for the triviality of D-brane charge in non-compactly supported K-theory (see footnote 6).

orbifolds are denoted $OF1_A$ and $OP1_A$, respectively, and the type-IIB orbifolds are denoted $OP1_B$ and $OF1_B$, respectively. The four theories are related by T-duality if we compactify some of the directions. In particular, $OF1_A$ is related to $OF1_B$ by T-duality of a coordinate along $\mathbb{R}^8$, and to $OP1_B$ by T-duality along the line. These orbifolds preserve 1/2 of the supersymmetry of type-II string theory. In particular, from the point of view of the invariant subspace $\mathbb{R}^{1,1}$, $OF1_{A,B}$ possess $\mathcal{N} = (8,8)$ supersymmetry, and $OP1_{A,B}$ have $\mathcal{N} = (16,0)$ (or $(0,16)$) supersymmetry.

3.1 Perturbative spectrum

We begin by describing the closed string spectrum of these theories. The *untwisted sector* is obtained simply by acting on the spectrum of type-II string theory by the orbifold projection

$$P = \frac{1}{2}(1+g)\,, \qquad (g = \mathcal{I}_8 \text{ or } \mathcal{I}_8(-1)^{F_L})\,. \tag{3.1}$$

Before orbifolding, the type-II spectrum in light-cone gauge consists of states created from a ground state in one of four sectors (NS-NS, NS-R, R-NS, and R-R) by left and right-moving oscillators of the form α^i_{-n}, ψ^i_{-n} (R), $\psi^i_{-n+1/2}$ (NS), where $n \geq 0$, and $i = 1,\ldots,8$ (the fixed line is taken to lie along the longitudinal coordinate x^9). The ground state of the NS sector is non-degenerate and tachyonic, with $m^2 = -1/2$, whereas the presence of 8 fermionic zero modes in the R sector give rise to a 16-fold degenerate massless ground state, i.e. a massless spinor. Consistency (and spacetime supersymmetry) requires that this spectrum be projected by $P_{GSO} \cdot \widetilde{P}_{GSO}$, where

$$P_{GSO} = \begin{cases} \frac{1}{2}(1+(-1)^f) \text{ NS} \\ \frac{1}{2}(1+(-1)^f) \text{ R} \end{cases} \qquad \widetilde{P}_{GSO} = \begin{cases} \frac{1}{2}(1+(-1)^{\tilde{f}}) \text{ NS} \\ \frac{1}{2}(1\pm(-1)^{\tilde{f}}) \text{ R}\,. \end{cases} \tag{3.2}$$

In the last line the $+$ sign corresponds to type IIB and the $-$ sign to type IIA. In particular, the projection on the R ground state reduces it to a chiral 8-component spinor (either $\mathbf{8}_s$ or $\mathbf{8}_c$ of SO(8)).

In the *twisted sector* the moding of the oscillators in the inverted directions changes, and becomes half-odd-integral for the bosons $\alpha^i_{-n+1/2}$ and Ramond fermions $\psi^i_{-n+1/2}$ (R), and integral for the Neveu-Schwarz fermions ψ^i_{-n} (NS). The ground state of the R sector, as usual, is massless, due to a cancellation between the bosonic and fermionic contributions. Now this state is a singlet of SO(8), since there are no fermionic zero modes. The ground state of the NS sector on the other hand is 16-fold degenerate, but is massive, with $m^2 = +1/2$. The only massless field is therefore a real RR scalar ϕ. Since the SO(1,1) chirality is fixed in light-cone gauge, this is actually a *chiral boson*.

A-priori this field exists in all four cases, but supersymmetry only allows it in the $\mathcal{N} = (16,0)$ theories. We therefore expect it to be removed from the other two theories by the GSO projection. This is most easily seen in a covariant gauge, where the $SO(1,1)$ chirality is not fixed. The R ground state is then a two-component Majorana spinor of $SO(1,1)$, which decomposes as $\mathbf{2} = \mathbf{1}_{+1/2} \oplus \mathbf{1}_{-1/2}$. The spectrum of fields in the twisted RR sector is then given by

$$\mathbf{2} \times \mathbf{2} = \mathbf{1}_{+1} \oplus \mathbf{1}_{-1} \oplus 2 \cdot \mathbf{1}_0 \,. \tag{3.3}$$

The first term corresponds to a chiral (self-dual) boson ϕ^+, and the second to an anti-chiral (anti-self-dual) boson ϕ^-. The third term is an unphysical state, and is removed in light-cone gauge. The GSO projection in the twisted sector of the $\mathcal{I}_8$ orbifold is the same as in the untwisted sector above, which means that the chiral and anti-chiral bosons are removed from type IIA, but one of them remains in type IIB. In the twisted sector of the $\mathcal{I}_8(-1)^{F_L}$ orbifold the left-moving GSO projection is reversed relative to the untwisted sector, so one of the bosons remains in type IIA, and both are removed from type IIB. We therefore find a massless chiral (or anti-chiral) RR boson only in the twisted sector of $OP1_{A,B}$. Finally, we must again project by (3.1), but this does not affect the above state.

The orbifolds $OF1_{A,B}$ do not have any massless propagating fields in the twisted sector. They may however possess non-dynamical fields, which are consistent with $\mathcal{N} = (8,8)$ supersymmetry. An example in ten dimensions is the RR 9-form potential in type-IIA string theory. This field is not dynamical, but is crucial for describing massive type-IIA backgrounds in which D8-branes are present. The analogous field in two dimensions is a vector potential A. We claim that precisely such a field exists in the twisted RR sector of the orbifolds $OF1_{A,B}$. In support of this claim we shall offer in the next subsection precisely the same argument which appeared in support of the 9-form in ten dimensions [18], namely the existence of a corresponding D-brane, which in our case is a (fractional) D-particle.

In the compact case, $\mathbf{T}^8/\mathbb{Z}_2$, there are additional fields corresponding to 16 fundamental strings, which must be added to cancel a one loop tadpole of the form $\int B \wedge Y_8(R)$ [11, 10] (see also [12]). This sets the NSNS charge (and tension) of a single bare orbifold line to $-1/16$. That the $OF1_B$ has the same $Q(0) = -1/16$ is a consequence of T duality (in some direction in the $\mathbf{T}^8$) which preserves both the number of orbifold lines and the number of fundamental strings needed to cancel the tadpole. This charge is also consistent with dualities: the $OF1_A$ lifts in M theory to the orbifold 2-plane (OM2), whose tension was determined to be $-1/16$ in [9]; the $OF1_B$ is S dual to the bare orientifold line which has tension $-1/16$ as well. Later we will find out that for the $OF1_A$ one gets an orbifold with zero tension by adding a D6-brane ($G^{(2)}$ torsion) to the bare (i.e. perturbative) orbifold. This lifts to a smooth orbifold in M-theory of the form $(\mathbb{R}^8 \times \mathbf{S}^1)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as a half-shift along the M-circle.

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3.2 D-branes

The analysis of the D-brane spectrum follows closely the boundary state techniques of [19, 20]. We will use the convention of defining a Dp-brane of type (r, s), where $r + s = p$, to have r spatial world-volume directions along the orbifold line,[10] and s along $\mathbb{R}^8$. The spectrum of D-branes on $\mathbb{Z}_2$ orbifolds contains two kinds of irreducible D-brane states: *fractional* D-branes and *truncated* D-branes. The former are linear combinations of boundary states in the untwisted NSNS and RR sectors, and the twisted NSNS and RR sectors,

$$|D(r,s)\rangle = \frac{1}{2}\Big(|B(r,s)\rangle_{\mathrm{NSNS}} + \epsilon_1|B(r,s)\rangle_{\mathrm{RR}} + \epsilon_1\epsilon_2|B(r,s)\rangle_{\mathrm{NSNS,T}} + \epsilon_2|B(r,s)\rangle_{\mathrm{RR,T}}\Big)\,, \tag{3.4}$$

and the latter contain only an untwisted NSNS state and a twisted RR state,

$$|\widetilde{D(r,s)}\rangle = \frac{1}{\sqrt{2}}\Big(|B(r,s)\rangle_{\mathrm{NSNS}} + \epsilon_2|B(r,s)\rangle_{\mathrm{RR,T}}\Big)\,. \tag{3.5}$$

The numerical factors in front are fixed by the requirement that, for any pair of D-branes, the cylinder amplitude corresponds to an open string partition function [21]. The signs ϵ_1, ϵ_2 determine the charges of the D-brane with respect to the appropriate massless RR field in the untwisted and twisted sector, respectively. The presence of twisted sector charge means that these D-branes are "stuck" to the orbifold line. By combining two fractional D-branes with opposite ϵ_2 one can form a *bulk* D-brane, which can separate from the orbifold line.

In a supersymmetric orbifold, such as the ones we are considering, fractional D-branes are BPS states, and truncated D-branes describe non-BPS states. The latter may or may not be stable, depending on whether their spectrum includes tachyons. In the non-compact case we are considering, truncated D-branes will only be stable for $s = 0$, i.e. if their world-volume is entirely transverse to the $\mathbb{R}^8$.

The spectrum of physical D-brane states is determined by requiring GSO and orbifold invariance. This has been done for the general case of n inverted directions in [20]. The results for $n = 8$ are summarized in table 3.

D-brane	$OF1_A$	$OP1_A$	$OP1_B$	$OF1_B$
$\lvert D(r,s)\rangle$	$r=0,\ s$ even	$r=\pm 1,\ s$ odd	$r=\pm 1,\ s$ even	$r=0,\ s$ odd
$\lvert\widetilde{D(r,s)}\rangle$	–	$r=\pm 1,\ s=0$	–	$r=s=0$

Table 3: Fractional and Truncated (D, and $\widetilde{D}$) D-brane spectrum in the background of the various orbifold lines. The D-brane has r directions parallel to the line, and s transverse to it.

[10] $r = -1$ corresponds to a Dirichlet boundary condition in time, i.e. a D-instanton.

In particular both $OF1_A$ and $OF1_B$ admit a D-particle with a twisted RR component. In the first case it is a BPS fractional D-particle, and in the second case it is a stable non-BPS truncated D-particle. This confirms the existence of a non-dynamical vector field A in the twisted sector of these theories.

In addition, $OF1_A$ contains fractional Dp-branes with $p = 2, 4, 6, 8$, which are transverse to the orbifold line and intersect it at a point. All of these are charged under A, as well as under the respective RR field $C^{(p+1)}$ in the untwisted sector. Similarly, $OF1_B$ contains transverse fractional Dp-branes with $p = 1, 3, 5, 7$, which are all charged under A, as well as the respective $C^{(p+1)}$. The two-dimensional scalar field strength $F = *dA$ can be thought of as a cosmological constant in the twisted sector, by analogy with the ten-dimensional cosmological constant $G^{(0)} = *dC^{(9)}$ in type-IIA string theory. The above fractional branes form domain walls on the $OF1$ orbifolds, across which the value of F jumps. The magnitude of the jump depends on the size of the fractional D-brane as follows:

$$\Delta F_{\mathrm{D}p} = 2^{-p/2}\,. \tag{3.6}$$

This is most easily determined by replacing $\mathbb{R}^8$ with the compact $\mathbf{T}^8$. In this case there are 2^8 fixed lines, each containing a RR 1-form in the twisted sector. The correctly normalized fractional D-brane states are (ignoring the volume factors) [20]

$$\begin{aligned} |D(r,s)\rangle = \frac{1}{2}\Bigg[& |B(r,s)\rangle_{\mathrm{NSNS}} + \epsilon_1 |B(r,s)\rangle_{\mathrm{RR}} + \\ & + 2^{-s/2} \sum_{i=1}^{2^s} \epsilon_2^{(i)} \Big(\epsilon_1 |B(r,s)\rangle_{\mathrm{NSNS,T_i}} + |B(r,s)\rangle_{\mathrm{RR,T_i}} \Big) \Bigg]\,, \end{aligned} \tag{3.7}$$

where i labels the orbifold lines which the D-brane intersects. Equation (3.6) follows by noting that $s = p$ for the $OF1$ fractional D-branes. In particular, for the fractional D-particle in type IIA the jump is $\Delta F = 1$. On the other hand, for the truncated D-particle in type IIB the jump is

$$\Delta F_{\widetilde{D0}} = \sqrt{2}\,. \tag{3.8}$$

This follows by comparing its boundary state (3.5) to that of the type-IIA fractional D-particle (3.4). Alternatively, this also follows from the realization of the non-BPS D-particle as a bound state of two BPS D-strings with opposite untwisted RR charge and equal twisted RR charge [22].

4. Orbifold line tension and NSNS charge

The perturbative orbifold lines (with $F = 0$) carry both an NSNS charge and tension given by $Q(0) = -1/16$. That the two are equal is guaranteed by the fact that the orbifold lines are BPS objects, i.e. they preserve 1/2 of the underlying supersymmetry.[11] This will continue to be true for non-zero values of F, since F does not have a fermionic partner, and therefore does not appear in the SUSY variation equations. As we shall see, the tension, and therefore the charge, depends on the value of F.

4.1 Effective action

The low-energy effective action of the orbifold theories contains a standard kinetic term for the twisted sector vector field

$$S_F = -\frac{n}{2}\int F^2\,, \tag{4.1}$$

where the coefficient n remains to be determined, though a 1/2 prefactor is inserted with hindsight. This implies that the tension jump due to a Dp-brane is given by $\Delta Q_{\mathrm{D}p} = (n/2)\Delta F^2_{\mathrm{D}p}$. The normalization factor n could a-priori be different for IIA and IIB, but T-duality actually shows it is the same. Consider a fractional Dp-brane in IIA intersecting $\mathbf{T}^8/\mathbb{Z}_2$. Since a fractional D$p$-brane intersects 2^p orbifold lines, the total tension jump is $\Delta Q = 2^p(n_A/2)\Delta F^2_{\mathrm{D}p} = n_A/2$. After T-duality the same computation yields $\Delta Q = n_B/2$, but since tension is invariant under T-duality we have $n_A = n_B$.

To determine n we could in principle compute the closed string sphere amplitude with two insertions of F. Alternatively, it can also be deduced from the cylinder amplitude involving the twisted RR component of a fractional D-particle,

$$A_{\mathrm{RR,T}} = \frac{1}{4}\langle B0|e^{-lH_{\mathrm{closed}}}|B0\rangle_{\mathrm{RR,T}}\,, \tag{4.2}$$

which in the limit $l \to \infty$ is equivalent to the sphere amplitude. On the other hand, the cylinder amplitude can be re-expressed as a one loop open string amplitude, which in the above limit reduces to the supersymmetry index of the internal CFT, giving

$$n = -\mathrm{Tr}^{0-0}_{\mathrm{NS-R}}(-1)^f\left(\frac{1+g}{2}\right) = 8\,. \tag{4.3}$$

The computation proceeds as follows. There are 8 massless states in the NS sector $\psi^i_{-1/2}|0\rangle_{\mathrm{NS}}$ prior to the projection, where the index $i = 1,\ldots,8$ is transverse to the $OF1_A$. However since $\psi^i_{-1/2}$ is odd under the orbifold projection, all these states are projected out, and no spacetime bosons remain. In the R sector there are 8 fermionic zero modes ψ^i_0, which upon quantization yield 16 massless states. The action of g on the R ground state is given by $\prod_i \psi^i_0$, i.e. it is the same as the action of $(-1)^f$ on the R ground state in the internal CFT, so the only non-vanishing contribution comes from $\frac{1}{2}\mathrm{Tr}_R(-1)^f g = \frac{1}{2}\mathrm{Tr}_R\mathbf{1} = 8$.

[11] More precisely they preserve 16 supersymmetries.

The tension jump is therefore given by

$$\Delta Q_{\mathrm{D}p} = 4\Delta F_{\mathrm{D}p}^2 = \begin{cases} 2^{-p+2} & \text{for a fractional D}p\text{-brane} \\ 8 & \text{for the non-BPS D-particle}\,. \end{cases} \tag{4.4}$$

Taking into account that $Q(0) = -1/16$ the absolute tension (and NSNS charge) as a function of the background RR field F is then given by

$$Q(F) = 4F^2 - \frac{1}{16}\,. \tag{4.5}$$

4.2 String creation

A physical picture for the tension and charge jumps (and our original derivation of the above formula) is provided by the string creation phenomenon. This phenomenon is an example of the Hanany-Witten effect [13], in which two linked D-branes, i.e. two D-branes which have a single mutually transverse direction, cross each other [14, 15]. In order to conserve the linking number, a string must be created between the two D-branes.

An example of a linked pair is the D0-D8 system in type IIA. From the D8-brane point of view, the linking number corresponds to a world-volume electric charge induced by the D-particle, which flips sign when the branes cross. Similarly, the D8-brane induces a charge on the D-particle, which flips sign when the branes cross. A unique feature of this particular linked brane system is that, since the D-particle cannot support a net charge, the induced charge must be cancelled by strings ending on it. For a single D8-brane this requires an unusual $\pm 1/2$ string (depending on which side of the D8-brane the D-particle sits), so the corresponding system is presumably unphysical (a system with two D8-branes is more physical), but the net change when the branes cross is the creation of a single string.

The process of string creation can also be understood dynamically. The ground state of the D0-D8 open string is a single massless fermion, which leads to a repulsive one-loop effective potential in the world-line theory of the D-particle,[12]

$$V(x) = -|x|/2\,. \tag{4.6}$$

This gives a jump in the force at $x = 0$, which is exactly compensated by the creation of a single string between the D-particle and the D8-brane. More generally, the number of strings which are created is given by the number of (massless) R ground states of the open string between the two D-branes in the linked system. This is equivalent to the supersymmetry index of the open string, since the NS ground state is always massive in such a system.

[12]The full string effective potential vanishes by supersymmetry. From the D-particle point of view there exists a tree-level term which cancels the one-loop potential [15]. The former can be understood as coming from the (1/2) string attached to the D-particle.

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In our orbifold lines any two fractional D-branes are linked, and therefore exhibit string creation. In particular, for a pair of fractional D-particles on $OF1_A$ there are $n = 8$ massless R ground states, and therefore 8 strings are created.

We will now use this fact to compute the tension jump due to a single D-particle. The two D-particles divide the orbifold line into three regions (see figure 1): a semi-infinite line with $F-1$ field strength, a finite segment with F, and another semi-infinite line with $F+1$. We denote the corresponding tensions (or charges) $Q(F-1), Q(F)$ and $Q(F+1)$, respectively. Next, we exchange the positions of the two D-particles, and demand that $Q(F)$ remains unchanged in the middle segment,

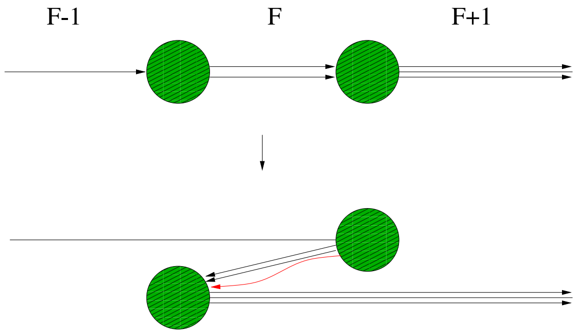

Figure 1: String creation when two D0 branes are crossing on an orbifold line. The wavy line is the newly created string.

$$Q(F) = Q(F-1) - Q(F) + Q(F+1) - 8\,. \tag{4.7}$$

The general solution to this recursion equation is $Q(F) = 4F^2 + aF + Q(0)$, which is consistent with (4.5).

To determine the value of the constant a requires knowledge of how many strings are attached to the D-particle to begin with, and not just the number of strings which are created in the exchange. As in the D0-D8 system, this is given by (minus) the electric charge induced on a D-particle by a background field F. By analogy with the D0-D8 system, we expect this charge to be of the form $8 \cdot F$. The situation here is somewhat more subtle, in that the D-particle is both a source and a probe for the field F. It is therefore not clear what should be taken as the background value of F. A natural choice in the above case of the two D-particles is the average of the two sides, $F + 1/2$. The net number of strings attached to the probe D-particle is then given by $Q(F+1) - Q(F) = 8 \cdot (F + 1/2)$, which gives $Q(F) = 4F^2 + Q(0)$.

4.3 The self intersection number

Here we offer an alternative point of view for the computation of $\Delta Q(F)$ which uses the self intersection in K-theory. Type IIA includes the following coupling[13]

$$\frac{1}{2}\int B_{\rm NS} \wedge G^{(4)} \wedge G^{(4)}\,, \tag{4.8}$$

where $G^{(4)}$ is the 4-form RR field strength, and all forms are normalized to have integral fluxes. This term contributes $\Delta Q = (1/2)\int G^{(4)} \wedge G^{(4)}$ to the fundamental string

[13] We thank E. Witten for suggesting this. The normalization is inherited from the term $1/6\int C \wedge G \wedge G$ in 11d.

charge. We would like to perform this integral by using a Poincaré duality to exchange $G^{(4)}$ with the dual 4-cycle, and trade the wedge product with the intersection product. However, since $\mathbb{R}^8/\mathcal{I}_8$ is singular, one needs to be careful.

The eight-dimensional orbifold $\mathbb{R}^8/\mathcal{I}_8$, unlike $\mathbb{R}^4/\mathcal{I}_4$, cannot be resolved in the usual manner. The necessary "blow-up" moduli, which appear as massless scalars in the twisted NSNS sector of $\mathbb{R}^4/\mathcal{I}_4$, are absent for $\mathbb{R}^8/\mathcal{I}_8$. The twisted NSNS sector of the orbifold line is purely massive. This orbifold therefore corresponds to an isolated point in the moduli space of type-II compactifications to two dimensions, i.e. it is not a limit of Calabi-Yau four-folds.

Nevertheless, one expects it to correspond to a limit of some smooth eight-dimensional manifold X, which is not Ricci-flat. Since the only massless field in the twisted sector of this orbifold for type IIA ($OF1_A$) is a RR vector, we expect the above eight-dimensional manifold to possess a single homology cycle C of dimension four. The IIA vector then corresponds to the reduction of $C^{(5)}$ on the four-cycle.

Assuming that such a manifold X exists, and that it has a 4 cycle C, we can take the dual of $G^{(4)}$ for the orbifold with $F = 1$ to be C. From (4.8) we then find $\Delta Q(F) = (1/2)(F \cdot C) \circ (F \cdot C) = (1/2)(C \circ C)F^2$, from which we deduce

$$|(C \circ C)| = n = 8. \tag{4.9}$$

To be more precise one should replace the cohomological formula (4.8) by a K-theoretic one $\Delta Q(F) = (1/2)(F \cdot y) \circ (F \cdot y)$, where y is the class of the D-particle, i.e. the generator of $K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{8,0})$, and $y \circ y$ is a K-theory product. It follows from (2.8) that $y \circ y = 8$.

We should point out, however, that so far we have not been able to identify the manifold X. We found two manifolds X satisfying $\partial X = \mathbb{RP}^7$, but neither one has the right properties. Presumably these are not the right manifolds that can be continuously deformed (continuous with respect to K-theory data) into the singular $\mathbb{C}^4/\mathbb{Z}_2$. The first, X_1, is the disk bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^3$, which was discussed in [9]. It is gotten from the circle fibration $\mathbf{S}^1 \to \mathbb{RP}^7 \to \mathbb{CP}^3$, by filling the $\mathbf{S}^1$. It does have a 4 cycle C, but it also has a redundant 2 cycle and a redundant 6 cycle. Moreover, the self intersection number of C is -2. Another space, X_2, can be gotten from the fibration $\mathbf{S}^3 \to \mathbf{S}^7 \to \mathbf{S}^4$, which induces $\mathbb{RP}^3 \to \mathbb{RP}^7 \to \mathbf{S}^4$. By filling the fiber with an Eguchi-Hanson space (EH) one gets EH $\to X_2 \to \mathbf{S}^4$. However, X_2 inherits a redundant 2 cycle and a redundant 6 cycle from the 2 cycle of EH. Thus neither X_1 nor X_2 meets our expectations.

A. An orbifold K-theory primer

A.1 Equivariant K-theory

Consider a space X acted on by a discrete symmetry group G. A vector bundle E over X, acted on by G such that the projection $E \to X$ commutes with the action of G, is called a G-*equivariant* bundle over X/G. Pairs of such bundles (E,F), modulo the usual identification $(E\oplus H, F\oplus H)$ (where H is also a G-equivariant bundle), are classified in equivariant K-theory $K_G(X)$. These groups satisfy many of the properties of ordinary K-theory, such as Bott-periodicity,

$$K_G^{-k}(X) = K_G^{-k+2}(X)\,. \tag{A.1}$$

Two basic identities in equivariant K-theory which we will use are

$$K_G(pt.) = R[G]\,, \tag{A.2}$$

where $R[G]$ is the representation ring of G, and

$$K_G(X) = K(X/G) \tag{A.3}$$

if G acts *freely* on X. As in ordinary K-theory, one can also define the notion of *reduced* equivariant K-theory, $\widetilde{K}_G(X)$, by requiring E and F to have equal rank, or equivalently by removing the group evaluated at a (G-orbit of a) generic point,

$$\widetilde{K}_G(X) = \ker\left[K_G(X) \to K_G(\cup_i g_i\ pt.)\right] \qquad (g_i \in G)\,. \tag{A.4}$$

Using (A.3) we see that the latter group is simply $K(pt.) = \mathbb{Z}$, so the usual decomposition holds,

$$K_G(X) = \widetilde{K}_G(X) \oplus \mathbb{Z}\,. \tag{A.5}$$

Let $X = \mathbb{R}^n$ and $G = \mathbb{Z}_2$, where the non-trivial element inverts l of the n coordinates. The corresponding group is denoted $K_{\mathbb{Z}_2}(\mathbb{R}^{l,m})$, where $m = n - l$. Since $\mathbb{R}^n$ is (equivariantly) contractable to a point,

$$K_{\mathbb{Z}_2}(\mathbb{R}^{l,m}) = R[\mathbb{Z}_2] = \mathbb{Z}\oplus\mathbb{Z}\,. \tag{A.6}$$

If the bundles E and F are isomorphic at infinity, the pair (E,F) is said to be *compactly supported*, and the classifying groups are given by [23]

$$K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{l,m}) = K_{\mathbb{Z}_2}(\mathbb{R}^{l,m}, \mathbf{S}^{l,m}) = \begin{cases} 0 & m \ \text{odd} \\ \mathbb{Z} & m \ \text{even},\ l \ \text{odd} \\ \mathbb{Z}\oplus\mathbb{Z} & m, l \ \text{even}\,, \end{cases} \tag{A.7}$$

where $\mathbf{S}^{l,m}$ denotes the $(n-1)$-sphere (at infinity) in the space $\mathbb{R}^n/\mathbb{Z}_2$. There is no further reduction of these groups, as the compact support condition already implies that E and F have equal rank. Higher K-theory groups are obtained by the suspension isomorphism,

$$K^{-k}_{\mathbb{Z}_2,cpct}(\mathbb{R}^{l,m}) = K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{l,m+k})\,, \tag{A.8}$$

and are related by the usual Bott-periodicity (A.1). In particular this implies that

$$K^{-1}_{\mathbb{Z}_2}(\mathbb{R}^{l,m}) = K^{-1}_{\mathbb{Z}_2}(pt.) = K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{0,1}) = 0\,. \tag{A.9}$$

The computation of equivariant K-theory groups for the spheres $X = \mathbf{S}^n$ is somewhat more involved in general. However, for an $(n-1)$-sphere corresponding to the boundary of the completely reflected space $\mathbb{R}^{n,0}$, denoted $\mathbf{S}^{n,0}$, the action of $\mathbb{Z}_2$ is free, and we can use (A.3) to get [24]

$$K_{\mathbb{Z}_2}(\mathbf{S}^{n,0}) = K(\mathbb{RP}^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}_{2^{[(n-1)/2]}}\,. \tag{A.10}$$

The discrete torsion part corresponds to the reduced group $\widetilde{K}(\mathbb{RP}^{n-1})$, and can be determined as follows. The generator of $\widetilde{K}(\mathbb{RP}^{n-1})$ is $(\mathcal{L}_{\mathbb{C}}, \mathbf{1})$, where $\mathcal{L}_{\mathbb{C}}$ is the complexification of the canonical (real) line bundle $\mathcal{L}$ over $\mathbb{RP}^{n-1}$. Recall that a point p in $\mathbb{RP}^{n-1}$ corresponds to a real line through the origin in $\mathbb{R}^n$. The fiber of $\mathcal{L}$ above the point p is defined to be precisely this line. The line bundle can also be expressed as $(\mathbf{S}^{n-1} \times \mathbb{R})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ generator simultaneously reflects the sphere and the line. (For example, the canonical line bundle over $\mathbb{RP}^1 \simeq \mathbf{S}^1$ is the Möbius strip.) One can, as usual, define local frames over subsets U_i of $\mathbb{RP}^{n-1}$ which trivialize the bundle locally, but the $\mathbb{Z}_2$ twist prevents these frames from combining into a global non-degenerate frame. Such a frame can however be constructed by considering multiple copies of the bundle, and is given by $\Gamma^i \cdot \widehat{x}^i$, where $i = 1, \ldots, n$. This requires $2^{[(n-1)/2]}$ copies of $\mathcal{L}_{\mathbb{C}}$, since that is the dimension of the irreducible complex spinor representation of $\mathrm{SO}(n)$. (Note that for $n = 2$, $\mathcal{L}_{\mathbb{C}}$ is twice the real Möbius bundle, and is therefore trivial.)

The different groups are part of a long exact sequence given by

$$\begin{aligned} \cdots \to K^{-1}_{\mathbb{Z}_2}(\mathbb{R}^{l,m}) \to K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{l,m}) \to K_{\mathbb{Z}_2,cpct}(\mathbb{R}^{l,m}) & \\ \xrightarrow{i} \widetilde{K}_{\mathbb{Z}_2}(\mathbb{R}^{l,m}) \to \widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{l,m}) \to K^{-1}_{\mathbb{Z}_2,cpct}(\mathbb{R}^{l,m}) \to \cdots\,, & \end{aligned} \tag{A.11}$$

which, for $l = n$ and $m = 0$ reduces to

$$0 \to K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{n,0}) \to \mathbb{Z}' \oplus \mathbb{Z} \xrightarrow{i} \mathbb{Z} \to \mathbb{Z}_{2^{[(n-1)/2]}} \to 0\,. \tag{A.12}$$

It follows that $K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{n,0}) = \mathbb{Z}'$, and that the map i is a multiplication by $2^{[(n-1)/2]}$.

A.2 Atiyah-Hirzebruch spectral sequence

There is a useful technique of successive approximations to K-theory known as the Atiyah-Hirzebruch spectral sequence (AHSS). We start by triangulating the n-dimensional space X with successive trees X^p, where $p = 0, \ldots, n$, and forming the filtration

$$K_n(X) \subseteq K_{n-1}(X) \subseteq \cdots \subseteq K_0(X) = K(X)\,, \tag{A.13}$$

where $K_p(X)$ is defined to contain classes which are trivial on the $(p-1)$-tree X^{p-1}. For example $K_1(X) = \widetilde{K}(X)$. Analogous filtrations can be formed for $K^{-1}(X)$, and for equivariant K-theory. The second step is to define a graded complex associated to the K-theory group in question, e.g.

$$\mathrm{Gr}K^{-k}(X) = \oplus_p K_p^{-k}(X)/K_{p+1}^{-k}(X)\,. \tag{A.14}$$

The successive ratios forming the graded complex are computed using a spectral sequence of groups $E_r^{p,q}$ with differential maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, such that each $E_{r+1}^{p,q}$ is the cohomology of d_r. The first terms in the sequence $E_1^{p,q}$ are given by singular p-cochains on X with values in $K^{-q}(pt.)$,

$$E_1^{p,q} = C^p(X; K^{-q}(pt.)) = \begin{cases} C^p(X;\mathbb{Z}) & q \text{ even} \\ 0 & q \text{ odd}\,. \end{cases} \tag{A.15}$$

The differential d_1 can be identified with the simplicial co-boundary operator acting on these co-chains. The second set of terms in the sequence is therefore given by

$$E_2^{p,q} = H^p(X, K^{-q}(pt.)) = \begin{cases} H^p(X;\mathbb{Z}) & q \text{ even} \\ 0 & q \text{ odd}\,. \end{cases} \tag{A.16}$$

The successive ratios are given by the limit of this sequence,

$$K_p^{-k}(X)/K_{p+1}^{-k}(X) = E_\infty^{p,-(p+k)}\,, \tag{A.17}$$

although in the cases we are interested in the sequence terminates at $E_2^{p,q}$. For example, for $\mathbf{S}^{8,0}$ we get

$$\begin{aligned} \mathrm{Gr}K_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) &= H^{\text{even}}(\mathbb{RP}^7;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2 \\ \mathrm{Gr}K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) &= H^{\text{odd}}(\mathbb{RP}^7;\mathbb{Z}) = \mathbb{Z}\,. \end{aligned} \tag{A.18}$$

Once we have the graded complex, computing the filter groups K_p^{-k}, and in particular the K-theory group itself, requires a solution of extension problems of the form

$$0 \to K_{p+1}^{-k}(X) \to K_p^{-k}(X) \to K_p^{-k}(X)/K_{p+1}^{-k}(X) \to 0\,. \tag{A.19}$$

In the second case the solution is unique, and gives

$$K^{-1}_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) = \mathbb{Z}\,, \tag{A.20}$$

but in the first case there are three possible solutions: $\mathbb{Z}\oplus\mathbb{Z}_2^3$, $\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4$, and $\mathbb{Z}\oplus\mathbb{Z}_8$. From the previous discussion we know that the correct answer is $K_{\mathbb{Z}_2}(\mathbf{S}^{8,0}) = \mathbb{Z}\oplus\mathbb{Z}_8$.

Although the extension problem precludes the AHSS as a useful way of computing the latter group, it does offer some physical insight into the nature of the discrete torsion "enhancement" in K-theory compared to cohomology. The point is that the

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filter groups K_p ($p \geq 1$) can also be thought of as K-theory groups which classify RR fields starting with a D$(9-p)$-$\overline{\text{D}(9-p)}$-brane pair. The filter groups for $K_{\mathbb{Z}_2}(\mathbf{S}^{8,0})$ are therefore given by

$$K_p(\mathbf{S}^{8,0}) = \widetilde{K}_{\mathbb{Z}_2}(\mathbf{S}^{9-p,0})\,. \tag{A.21}$$

Using (A.10) we get

$$\begin{aligned} K_1 &= K_2 = \mathbb{Z}_8 \\ K_3 &= K_4 = \mathbb{Z}_4 \\ K_5 &= K_6 = \mathbb{Z}_2 = H^6(\mathbb{RP}^7;\mathbb{Z}) \\ K_7 &= 0\,, \end{aligned} \tag{A.22}$$

where the identification of the $\mathbb{Z}_2$ above with H^6 follows from (A.16), (A.17), and the fact that $K_7 = 0$. Consider now the two sequences of the form (A.19) with $p = 4$ and $p = 2$,

$$\begin{array}{ccccccccc} 0 & \to & K_5 & \to & K_4 & \to & H^4(\mathbb{RP}^7;\mathbb{Z}) & \to & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}_2 & \to & \mathbb{Z}_4 & \to & \mathbb{Z}_2 & & \\ & & & & & & & & \\ 0 & \to & K_3 & \to & K_2 & \to & H^2(\mathbb{RP}^7;\mathbb{Z}) & \to & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}_4 & \to & \mathbb{Z}_8 & \to & \mathbb{Z}_2 & & \end{array} \tag{A.23}$$

Denote the generator of $\mathbb{Z}_8$ by x, the generator of $\mathbb{Z}_4$ by y, and the generators of the three even ($\mathbb{Z}_2$) cohomologies $x^{(2)}, x^{(4)}$, and $x^{(6)}$. It then follows from the above sequences that

$$x = x^{(2)}\,,\ 2x = x^{(4)}\,,\ 4x = x^{(6)}\,, \tag{A.24}$$

which gives equation (2.3).

A.3 Hopkins' groups $K_\pm$

If the generator of the $\mathbb{Z}_2$ includes the operator $(-1)^{F_L}$ in addition to inversion, the bundles are not equivariant, but rather we are given an isomorphism $\lambda : (E, F) \to (g^*F, g^*E)$ with $(\lambda g^*)^2 = 1$. The corresponding groups are denoted $K_\pm(X)$. These groups have not been studied as extensively as equivariant K-theory, so far less is known about them (see [25] for related results).

A theorem due to Hopkins (unpublished) asserts that

$$K_{\pm,cpct}(X) = K_{\mathbb{Z}_2,cpct}(X \times \mathbb{R}^{1,1})\,. \tag{A.25}$$

This relation can be understood as the failure of ordinary Bott periodicity $K_{cpct}(X) = K_{cpct}(X \times \mathbb{R}^2)$, due to the orientation reversal of the $\mathbb{R}^2$.[14] It should be contrasted

[14]This was pointed out to us by M. Atiyah.

with the case in which the orientation is preserved,

$$K_{\mathbb{Z}_2,cpct}(X) = K_{\mathbb{Z}_2,cpct}(X \times \mathbb{R}^{0,2})\,. \tag{A.26}$$

In particular this implies

$$\begin{aligned} K_\pm(\mathbb{R}^{l,m}) \ &= K_\pm(pt.) \ &= K_{\pm,cpct}(\mathbb{R}^{0,0}) &= \ 0 \\ K_\pm^{-1}(\mathbb{R}^{l,m}) &= K_\pm^{-1}(pt.) &= K_{\pm,cpct}^{-1}(\mathbb{R}^{0,0}) &= \mathbb{Z}\,, \end{aligned} \tag{A.27}$$

where in the last step we used the suspension isomorphism (A.8), which continues to hold for $K_\pm$.

For the spheres, one can again try the AHSS. The analysis is very similar to the one above for equivariant K-theory, except that the singular cochains and cohomologies valued in $\mathbb{Z}$ are replaced by those valued in the twisted $\widetilde{\mathbb{Z}}$. This is because all RR fields are odd under $(-1)^{F_L}$. The corresponding graded complexes are therefore given by

$$\begin{aligned} \mathrm{Gr}K_\pm(\mathbf{S}^{8,0}) \ &= H^{\mathrm{even}}(\mathbb{RP}^7;\widetilde{\mathbb{Z}}) = 0 \\ \mathrm{Gr}K_\pm^{-1}(\mathbf{S}^{8,0}) &= H^{\mathrm{odd}}(\mathbb{RP}^7;\widetilde{\mathbb{Z}}) \ = \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\,. \end{aligned} \tag{A.28}$$

In particular, this implies that $K_\pm(\mathbf{S}^{8,0}) = 0$. The second group $K_\pm^{-1}(\mathbf{S}^{8,0})$ suffers from the same extension problems as $K_{\mathbb{Z}_2}(\mathbf{S}^{8,0})$, and it is reasonable to guess that a complete enhancement occurs to $\mathbb{Z}_{16}$. From (A.25) it follows that

$$K_\pm^{-1}(\mathbf{S}^{n,0}) = K^{-1}_{\mathbb{Z}_2,cpct}(\mathbf{S}^{n,0}\times\mathbb{R}^{1,1}) = K_{\mathbb{Z}_2,cpct}(\mathbf{S}^{n,0}\times\mathbb{R}^{1,0})\,, \tag{A.29}$$

where in the second equality we have used the suspension isomorphism and Bott periodicity. The action on the space $X = \mathbf{S}^{n,0}\times\mathbb{R}^{1,0}$ is free, and gives $X/\mathbb{Z}_2 \sim \mathcal{L}_{n-1}$, where $\mathcal{L}_{n-1}$ is the canonical real line bundle over $\mathbb{RP}^{n-1}$. Since the compactification of $\mathcal{L}_{n-1}$ is topologically the same as $\mathbb{RP}^n$ we find

$$K_\pm^{-1}(\mathbf{S}^{n,0}) = \widetilde{K}(\mathbb{RP}^n) = \mathbb{Z}_{2^{[n/2]}}\,, \tag{A.30}$$

and in particular $K_\pm^{-1}(\mathbf{S}^{8,0}) = \mathbb{Z}_{16}$. Similarly, one can show that $K_\pm(\mathbf{S}^{n,0}) = 0$.[15]

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